PHYSICALLY ADMISSIBLE CROSSED PRODUCTS.

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ABSTRACT. Not every crossed product $\mathcal{M} \rtimes_{\alpha} \mathbb{R}$ is a physically admissible algebra.

FIGURE 1. Ough.

CONTENTS

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1. Introduction

Crossed products occur naturally in recent operator algebraic formulations of semiclassical gravity. Given a von Neumann algebra $\mathcal{M}$ and a continuous one-parameter automorphism group, the crossed product $\mathcal{M} \rtimes_\alpha \mathbb{R}$ adjoins unitaries that implement α. In modular theory, this takes a type III algebra into a semifinite algebra (e.g. turning a type III_1 algebra into a type II_∞ algebra in AdS microcanonical ensemble). Usually this additional group is constructed via a clock, an observer or an asymptotic charge. The point we wish to make, however, is that this is purely algebraic and does not tell you which ones are physically admissible. Given a pair $(\mathcal{M}, \alpha)$, one can always choose some normal faithful representation of $\mathcal{M}$ and use $L^2(\mathbb{R})$ to obtain a crossed product, and consequently these won't all be physical algebras that satisfy gravitational constraints. In this sense, we emphasize that in order for a crossed product to be a *physically admissible* algebra of Dirac observables the generators of the crossed product must be commute with the residual constraint. The importance of this observation is that it shows that the slow-roll type II_∞ algebra from Chen-Penington in de Sitter space cannot be a crossed product, since that by itself does not show that the canonical inflaton pair (ϕ, P_ϕ) deparametrizes the residual constraint.

In this note, we make this clear, by showing that a crossed product is physically admissible relative to a clock precisely when the constraint has a linear decomposition into $P_\Gamma + H$, where H acts on the non-clock factor.

2. Existence and admissibility

Definition 1. A **von Neumann pair** consists of $(\mathcal{M}, \alpha)$, where $\mathcal{M}$ is a von Neumann algebra and $\alpha : \mathbb{R} \to \mathrm{Aut}(M)$ is a group homomorphism such that $s \to \omega(\alpha_s(a))$ is continuous for all $a \in \mathcal{M}$ and every normal function $\omega \in \mathcal{M}$. Equivalently, α is pointwise ultraweakly continuous.

We wish to show that one can construct an arbitrary crossed product given $(\mathcal{M}, \alpha)$.

Proposition 1. Let $\rho : \mathcal{M} \to \mathcal{B}(\mathcal{H}_s)$ be a faithful normal representation. Then, for every $(\mathcal{M}, \alpha)$, define

$$(\pi_\alpha(a)\Psi)_\gamma = \rho(\alpha_{-\gamma}(a))\Psi_\gamma \; , \tag{1}$$

with the $-\gamma$ sign chosen for convenient covariance, and

$$(\lambda(s)\Psi)_\gamma = \Psi(\gamma - s) \; . \tag{2}$$

Then,

(1) π_α is a faithful normal representation of $\mathcal{M}$,
(2) λ is a strongly continuous unitary representation of $\mathbb{R}$, and
(3) $\lambda(s)\pi_\alpha(a)\lambda(s)^* = \pi_\alpha(\alpha_s(a))$.

The von Neumann crossed product

$$\mathcal{M} \rtimes_\alpha \mathbb{R} := (\pi_\alpha(\mathcal{M}) \cup \lambda(\mathbb{R}))'' \tag{3}$$

therefore exists for all $(\mathcal{M}, \alpha)$.

Proof. For $a \in \mathcal{M}$, since α is a von Neumann pair, the map $\gamma \mapsto \alpha_{-\gamma}(a)$ is continuous from $\mathbb{R}$ into $(\mathcal{M}, \sigma\text{-weak})$, and since ρ is normal it carries σ-weakly convergent bounded nets to weak-operator convergent nets, so

$$\gamma \mapsto \rho(\alpha_{-\gamma}(a)) \tag{4}$$

is weak-operator-topology continuous, hence weakly measurable and strongly measurable as an operator-valued field on $\mathbb{R}$.

Since each $\alpha_{-\gamma}$ is a $*$-automorphism, it is isometric, and since ρ is faithful it is isometric on $\mathcal{M}$, so that

$$||\rho(\alpha_{-\gamma}(a))|| = ||a|| \tag{5}$$

for all γ. Together with measurability, (5) shows that

$$\pi_\alpha(a) := \int_{\mathbb{R}}^{\oplus} \rho(\alpha_{-\gamma}(a))\, d\gamma \tag{6}$$

is a well-defined bounded decomposable operator on K as in (1), with $\|\pi\ _{alpha(a)\|=\|a\|}$.

Firstly, we can see that π_α is a unital $*$-homomorphism since[1]

$$(\pi_\alpha(ab)\Psi)_\gamma = \rho(\alpha_{-\gamma}(a)\alpha_{-\gamma}(b))\Psi_\gamma = \tag{7}$$

$$\rho(\alpha_{-\gamma}(a))\rho(\alpha_{-\gamma}(b))\Psi_\gamma = (\pi_\alpha(a)\pi_\alpha(b)\Psi)_\gamma \ \ , \tag{8}$$

$$(\pi_\alpha(a)^*\Psi)_\gamma = \rho(\alpha_{-\gamma}(a))^*\Psi_\gamma = \tag{9}$$

$$\rho(\alpha_{-\gamma}(a^*))\Psi_\gamma = (\pi_\alpha(a^*)\Psi)_\gamma \tag{10}$$

and $\rho(\alpha_{-\gamma}(\not\Vdash)) = \rho(1) = 1_{\mathcal{H}_s}$ for every γ, and so $\pi_\alpha(1) = 1_K$. Hence π_α is a unital $*$-homomorphism.

If $\pi_\alpha(a) = 0$, then since a decomposable operator vanishes if and only if its fiber vanishes for (almost every(VK- *??*)) γ, we have $\rho(\alpha_{-\gamma_0}(a)) = 0$ for some γ_0. Faithfulness of ρ then gives $\alpha_{-\gamma_0}(a) = 0$ and since $\alpha_{-\gamma_0}$ is injective, it being an automorphism, $a = 0$.

[1](VK- *Check + align better.*)

Let (a_n) be a norm bounded seq. in $\mathcal{M}$ with $a_n \to a$ σ (weakly). Since $\alpha_{-\gamma}$ and ρ are normal, $\rho(\alpha_{-\gamma}(a_n)) \to \rho(\alpha_{-\gamma}(a))$ in WOTfor each fixed γ. For $\Phi, \Psi \in K$ we have (VK- *double check.*)

$$\langle \Phi, \pi_\alpha(a_n)\Psi\rangle = \int_{\mathbb{R}} \langle \Phi_\gamma, \rho(\alpha_{-\gamma}(a_n))\Psi_\gamma\rangle \, d\gamma, \tag{11}$$

with the integrand converging pointwise in γ and dominated by $\big(\sup_n \|a_n\|\big)\|\Phi_\gamma\|\,\|\Psi_\gamma\| \in L^1(\mathbb{R}, d\gamma)$ by Cauchy–Schwarz, and π_α is σ-weakly continuous on bounded sets or normal.

Next, $\lambda(s)$ is a translation operator on $K = L^2(\mathbb{R}, d\gamma; \mathcal{H}_s)$. By translation-invariance of Lebesgue measure,

$$\|\lambda(s)\Psi\|^2 = \int \|\Psi_{\gamma-s}\|^2 \, d\gamma = \int \|\Psi_{\gamma'}\|^2 \, d\gamma' = \|\Psi\|^2, \tag{12}$$

so each $\lambda(s)$ is an isometric surjection of K. Directly,

$$(\lambda(s)\lambda(t)\Psi)_\gamma = (\lambda(t)\Psi)_{\gamma-s} = \Psi_{\gamma-s-t} = (\lambda(s+t)\Psi)_\gamma, \tag{13}$$

so $\lambda(s)\lambda(t) = \lambda(s+t)$ and $\lambda(0) = 1_K$; hence each $\lambda(s)$ is unitary with $\lambda(s)^* = \lambda(s)^{-1} = \lambda(-s)$ – that is,

$$(\lambda(s)^*\Psi)_\gamma = \Psi(\gamma+s)\ . \tag{14}$$

Strong continuity follows from uniform continuity. (VK- *on compactly supported functions...*)

Finally, fix $a \in \mathcal{M}$, $s \in \mathbb{R}$, and $\Psi \in K$, and set $\Phi = \lambda(s)^*\Psi$, so that $\Phi_\mu = \Psi_{\mu+s}$ for all μ, by (14). For every γ,

$$\begin{aligned}
(\lambda(s)\pi_\alpha(a)\lambda(s)^*\Psi)_\gamma &= (\pi_\alpha(a)\Phi)_{\gamma-s} &(15)\\
&= \rho(\alpha_{-(\gamma-s)}(a))\,\Phi_{\gamma-s} &(16)\\
&= \rho\big(\alpha_{s-\gamma}(a)\big)\,\Psi_{(\gamma-s)+s} &(17)\\
&= \rho\big(\alpha_{s-\gamma}(a)\big)\,\Psi_\gamma\ . &(18)
\end{aligned}$$

Since α is a homomorphism from $(\mathbb{R}, +)$, we have $\alpha_{-\gamma} \circ \alpha_s = \alpha_{s-\gamma}$, so $\rho(\alpha_{s-\gamma}(a)) = \rho(\alpha_{-\gamma}(\alpha_s(a)))$, giving

$$(\lambda(s)\pi_\alpha(a)\lambda(s)^*\Psi)_\gamma = \rho(\alpha_{-\gamma}(\alpha_s(a)))\Psi_\gamma = (\pi_\alpha(\alpha_s(a))\Psi)_\gamma \tag{19}$$

for every γ. Since γ was arbitrary,

$$\lambda(s)\pi_\alpha(a)\lambda(s)^* = \pi_\alpha(\alpha_s(a))\ , \tag{20}$$

which establishes part (3) of the proposition, and completes the proof. $\square$

Remark 1. None of this has a physical constraint imposed on $\mathcal{H}_s$ or the crossed product algebra. For example, one could choose $\alpha_s = \mathrm{id}_{\mathcal{M}}$, in which case $\pi_\alpha(a) = 1 \otimes \rho(a)$ and $\mathcal{M} \rtimes_\alpha \mathbb{R} \cong \mathcal{M} \bar{\otimes} L(\mathbb{R})$. Since $\mathbb{R}$ is abelian, the fourier transform identifies the group von Neumann algbera $L(\mathbb{R})$ with $L^\infty(\hat{R}) \cong L^\infty(\mathbb{R})$.

Remark 2. To put things into a cleaner perspective, pick an arbitrarily suitable self-adjoint operator k so that

$$\alpha_s(a) = \rho^{-1}\left(e^{isk}\rho(a)e^{-isk}\right) \ , \tag{21}$$

with its unitary group normalizing ρ. Unless k is actually related to a residual physical constraint of a physical clock, this does not define a physical flow and is just an arbitrary conjugation.

3. Physical admissibility

Definition 2. On $L^2(\mathbb{R}, d\gamma)$, let

$$(\Gamma\Psi)_\gamma = \gamma\Psi_\gamma \ , \tag{22}$$

$$(P\Psi)_\gamma = -i\frac{\partial\Psi}{\partial\gamma} \ , \tag{23}$$

with their standard self-adjoint realizations. We use the same symbols for $\Gamma \otimes 1$ and $P \otimes 1$ on

$$\mathcal{K} = L^2(\mathbb{R}, d\gamma) \otimes \mathcal{H}_s \ . \tag{24}$$

Their Weyl relation is

$$e^{itP}e^{iu\Gamma}e^{-itP} = e^{itu}e^{iu\Gamma} \ . \tag{25}$$

The Schrödinger representation is irreducible, so

$$\left(\{e^{itP}\}_{t\in\mathbb{R}} \cup \{e^{iu\Gamma}\}_{u\in\mathbb{R}}\right)'' = \mathcal{B}(L^2(\mathbb{R})) \otimes 1 \ , \tag{26}$$

and therefore

$$\left(\mathcal{B}(L^2(\mathbb{R})) \otimes 1\right)' = 1 \otimes \mathcal{B}(\mathcal{H}_s) \ . \tag{27}$$

By Schur's lemma it suffices to show the commutant of the generating set is scalar. If X commutes with every $e^{iu\Gamma}$, then since $\{e^{iu\Gamma}\}_{u\in\mathbb{R}}$ generates the maximal abelian algebra $L^\infty(\mathbb{R})$ of multiplication operators, $X = M_m$ for some $m \in L^\infty(\mathbb{R})$. If X also commutes with every e^{itP}, then $m(\gamma - t) = m(\gamma)$ for every t. This forces m to be (almost everywhere) constant. Hence $X \in \mathbb{C}1$, and the bicommutant is all of $\mathcal{B}(L^2(\mathbb{R})) \otimes 1$.

Let C be a self-adjoint residual constraint on $\mathcal{K}$ and let

$$U_C(t) = e^{itC} \ . \tag{28}$$

The bounded Dirac algebra is

$$\mathcal{D}_C := \{U_C(t) : t \in \mathbb{R}\}' . \tag{29}$$

Thus $X \in \mathcal{D}_C$ precisely when X is invariant under the constraint flow.

(VK- *Unbounded operators*)

Definition 3. (Exact clock.) The pair (Γ, P) is an *exact clock pair* for C when the constraint flow translates the clock coordinate at unit speed:

$$U_C(t)\, e^{iu\Gamma}\, U_C(t)^* = e^{itu}\, e^{iu\Gamma} \qquad (t, u \in \mathbb{R}) . \tag{30}$$

This is the Weyl form of $[C, \Gamma] = -i1$.

The regular construction previously singles out two types of generators: the represented copy $\pi_\alpha(\mathcal{M})$ and the unitaries $\lambda(s)$. To interpret this crossed product as an algebra of observables for the constrained system, these generators must be represented physically – that is, they must be tied to the constraint C and to the clock pair (Γ, P) fixed above.

Definition 4. (Physically admissible crossed product) Let $(\mathcal{M}, \alpha)$ be a von Neumann pair. A regular covariant representation (π, λ) of $(\mathcal{M}, \alpha)$ on $\mathcal{K}$ is *physically admissible* relative to (C, Γ, P) if:

(D1) (Γ, P) is an exact clock pair for C.
(D2) the implementing unitaries are the canonical clock translations,

$$\lambda(s) = e^{-isP} . \tag{31}$$

(D3) every crossed-product generator is a Dirac observable,

$$\pi(\mathcal{M}) \cup \lambda(\mathbb{R}) \subset \mathcal{D}_C . \tag{32}$$

(D4) the represented non-clock algebra acts at fixed clock value,

$$\pi(\mathcal{M}) \subset \{e^{iu\Gamma} : u \in \mathbb{R}\}' . \tag{33}$$

Remark 3. Physical admissibility is *not* a property of the factor $\mathcal{M} \rtimes_\alpha \mathbb{R}$ alone. It is a property of a specified covariant representation, together with the embedding of its canonical generators into the Dirac algebra. An arbitrary isomorphism like

$$\Phi : \mathcal{M} \rtimes_\alpha \mathbb{R} \longrightarrow \mathcal{A}_{\text{phys}} , \tag{34}$$

is insufficient unless the $\Phi(\pi_\alpha(a))$ and $\Phi(\lambda(s))$ satisfy (D2)–(D4).

We can now state the full physical admissibility requirement into a theorem.

Theorem 1. Assume (Γ, P) is an exact clock pair for a self-adjoint constraint C on

$$\mathcal{K} = L^2(\mathbb{R}) \otimes \mathcal{H}_s . \tag{35}$$

The following are equivalent:

(i) P strongly commutes with C;
(ii) there is a self-adjoint operator H on $\mathcal{H}_s$ such that

$$e^{itC} = e^{itP} \otimes e^{itH} \qquad (t \in \mathbb{R}); \tag{36}$$

(iii) the constraint deparametrizes as

$$C = P \otimes 1 + 1 \otimes H \ . \tag{37}$$

The operator H is identified uniquely above.

Proof. Assume (i). Strong commutation allows us to define a strongly continuous one-parameter unitary group

$$V(t) := e^{-itP} e^{itC} \ . \tag{38}$$

Using exact clock covariance (30) and the Weyl relation (25),

$$V(t)e^{iu\Gamma}V(t)^* = e^{-itP}\big(e^{itC}e^{iu\Gamma}e^{-itC}\big)e^{itP} \tag{39}$$

$$= e^{-itP}\big(e^{itu}e^{iu\Gamma}\big)e^{itP} \tag{40}$$

$$= e^{itu}e^{-itu}e^{iu\Gamma} = e^{iu\Gamma} \ . \tag{41}$$

Strong commutation of C and P also gives

$$V(t)e^{isP}V(t)^* = e^{isP} \ . \tag{42}$$

Hence $V(t)$ commutes with the complete clock Weyl algebra. By (26),

$$V(t) = 1 \otimes V_s(t) \tag{43}$$

for a unitary $V_s(t)$ on $\mathcal{H}_s$. The group law and strong continuity descend to V_s, so Stone's theorem gives a unique self-adjoint H satisfying $V_s(t) = e^{itH}$. Substituting into the definition of $V(t)$ yields (36), proving (ii).

The operators $P\otimes 1$ and $1\otimes H$ strongly commute, since they act on different tensor factors. The generator of the product group in (36) is therefore the self-adjoint closure of their algebraic sum. Uniqueness of Stone generators proves (iii).

Conversely, (iii) implies (36) by the joint functional calculus of $P \otimes 1$ and $1 \otimes H$. Since the summands act on different tensor factors, P strongly commutes with C. This proves (i), and the equivalence follows. □

The next theorem is the precise version of the physical-admissibility criterion.

Theorem 2. (Physical admissibility criterion) Let $(\mathcal{M}, \alpha)$ be a von Neumann pair. A physically admissible regular representation of $\mathcal{M} \rtimes_\alpha \mathbb{R}$ relative to (C, Γ, P) exists if and only if there are:

(a) a faithful normal representation $\rho : \mathcal{M} \to \mathcal{B}(\mathcal{H}_s)$, and

(b) a self-adjoint H on $\mathcal{H}_s$,

such that

$$C = P \otimes 1 + 1 \otimes H \;, \tag{44}$$

$$\rho(\alpha_s(a)) = e^{isH} \rho(a) e^{-isH} \qquad (a \in \mathcal{M}, s \in \mathbb{R}) \;. \tag{45}$$

In this case the physically admissible representation is necessarily, up to a unitary acting only on $\mathcal{H}_s$, the relational regular representation

$$\pi_H(a) = e^{-i\Gamma \otimes H}(1 \otimes \rho(a)) e^{i\Gamma \otimes H} \;, \tag{46}$$

$$\lambda_P(s) = e^{-isP} \;. \tag{47}$$

Proof. First assume (44) and (45). Let

$$W = e^{-i\Gamma \otimes H} \;. \tag{48}$$

And $(W\Psi)_\gamma = e^{-i\gamma H}\Psi_\gamma$. So

$$WPW^* = P \otimes 1 + 1 \otimes H = C \;. \tag{49}$$

Therefore

$$[C, \pi_H(a)] = W[P, 1 \otimes \rho(a)]W^* = 0 \tag{50}$$

in the sense of spectral commutation, so $\pi_H(a) \in \mathcal{D}_C$. Since P strongly commutes with C, $e^{-isP} \in \mathcal{D}_C$ as well. Moreover, W commutes with Γ, so $\pi_H(a)$ commutes with every spectral projection of Γ.

Concretely,

$$(\pi_H(a)\Psi)_\gamma = e^{-i\gamma H} \rho(a) e^{i\gamma H} \Psi_\gamma = \tag{51}$$

$$\rho(\alpha_{-\gamma}(a))\Psi_\gamma \;, \tag{52}$$

where the last equality follows from (45). Together with $(\lambda_P(s)\Psi)_\gamma = \Psi_{\gamma - s}$, this is exactly the regular covariant representation of Proposition **??**. Hence

$$\lambda_P(s)\pi_H(a)\lambda_P(s)^* = \pi_H(\alpha_s(a)) \;, \tag{53}$$

and all four conditions (D1)–(D4) hold.

Reversely, assume (π, λ) is physically admissible. By (D2) and (D3), $e^{-isP} \in \mathcal{D}_C$ for all s, so the spectral measures of P and C strongly commute. By (D1) and theorem 2, there is a self-adjoint H satisfying (44). Let $W = e^{-i\Gamma\otimes H}$, so $C = WPW^*$ by (49).

For $a \in \mathcal{M}$, set $X(a) \coloneqq W^*\pi(a)W$. Because $\pi(a) \in \mathcal{D}_C$, the operator $X(a)$ commutes with the spectral projections of P. By (D4), $\pi(a)$ commutes with the spectral projections of Γ, and W itself is decomposable over Γ; hence $X(a)$ also commutes with the spectral projections of Γ. Equation (27) then forces

$$X(a) = 1 \otimes \rho(a) \ , \tag{54}$$

For a unique bounded operator $\rho(a)$ on $\mathcal{H}_s$. The map ρ is a faithful normal $*$-representation because $a \to X(a)$ is obtained from the faithful normal representation π by conjugation.

Finally, a direct calculation (VK- *check!*) gives

$$W^* lambda(s) W = e^{-isP} \otimes e^{isH} \ . \tag{55}$$

Conjugating the covariance identity $\lambda(s)\pi(a)\lambda(s)^* = \pi(\alpha_s(a))$ by W^* and using that $W^*\pi(a)W = 1 \otimes \rho(a)$, we get

$$1 \otimes e^{isH} \rho(a) e^{-isH} = 1 \otimes \rho(\alpha_s(a)) \ . \tag{56}$$

Hence (45) holds. This proves necessity and completes the proof. □

Corollary 1. Suppose (Γ, P) is proposed as the physical clock pair. If P does not strongly commute with the residual constraint C, then no crossed product whose canonical implementing unitaries are e^{-isP} is physically admissible relative to that clock. Nevertheless, Proposition **??** still constructs the abstract algebra $\mathcal{M} \rtimes_\alpha \mathbb{R}$ for every action α.

Proof. Physical admissibility would put every e^{-isP} in $\mathcal{D}_C$, which is equivalent to strong commutation of P and C. The abstract existence statement is independent of C and follows from proposition **??**. □

Remark 4. On a common invariant core, Theorem 2 has the familiar form

$$[C, \Gamma] = -i1 \ , \qquad [C, P] = 0 \qquad \Longleftrightarrow \qquad C = P + H \ , \tag{57}$$

with H commuting with the full clock Weyl algebra. The Weyl-form proof is stronger because it avoids domain ambiguities and makes clear that the required commutation is strong commutation, not merely vanishing of a formal commutator on a small core.

4. In de Sitter

In Chen and Penington's paper on clocks and de Sitter space, they note that the type II_∞ algebra obtained from a slowly rolling inflation solution does not appear to be a manifestly crossed product algebra. This is made a little more clear by noting that such an algebra is not a physically admissible crossed product, since the boost generator has mode-mixing and does not give a linear deparametrization.

Appendix A. Proof of (55)

The identity rests on how Γ's Weyl unitaries conjugate P.

Lemma 2. For all $u \in \mathbb{R}$,

$$e^{iu\Gamma}\, P\, e^{-iu\Gamma} = P - u1\ . \tag{58}$$

Proof. On a core of smooth, compactly supported ψ, (VK- *Update later*)

□

References

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